§7.8 Repeated Eigenvalues ⇝ Using Generalized Eigenvectors

(for fun and profit)

~~(These notes contain many extra examples that were not given in lecture)~~

Plan: Compute e^{At} for repeated eigenvalue A

→ Work the same as in lecture 10.

Baby example:

$\underline{x}' = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \underline{x}$ ⇝

Eigenvalues $\boxed{\lambda, \lambda}$

λ-eigenvect

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{v} = \underline{0}$ ⇝ $\boxed{\underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$-gen. eigenvect

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ⇝ $\boxed{\underline{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

???

alternate soln

$$\begin{cases} x' = \lambda x + y \\ y' = \lambda y \implies y = c_2 e^{\lambda t} \end{cases}$$

$x' = \lambda x + c_2 e^{\lambda t}$

Linear DE: $\mu = e^{-\lambda t}$

$e^{-\lambda t} x = \int c_2 \cancel{e^{\lambda t} \cdot e^{-\lambda t}}^{1}\, dt$

$= c_2 t + c_1$

$x = c_2 t e^{\lambda t} + c_1 e^{\lambda t}$

⇝ General Solution:

$$\underline{x} = c_1 e^{\lambda t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{\lambda t} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Note: $\begin{bmatrix} t \\ 1 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

eigenvect — gen. eigenv.

Guess: $e^{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} t} \overset{???}{=} \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$

Let's look at some power series...

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^2 = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^3 = \begin{bmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{bmatrix}$$

etc...

So $\left(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} t \right)^n = \begin{bmatrix} (\lambda t)^n & nt \cdot (\lambda t)^{n-1} \\ 0 & (\lambda t)^n \end{bmatrix}$

$$e^{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} t} = \begin{bmatrix} 1 + (\lambda t) + \frac{1}{2}(\lambda t)^2 + \frac{1}{3!}(\lambda t)^3 + \cdots & 0 + t + \frac{1}{2} 2t(\lambda t) + \frac{1}{3!} 3t(\lambda t)^3 + \cdots \\ 0 & 1 + (\lambda t) + \frac{1}{2}(\lambda t)^2 + \frac{1}{3!}(\lambda t)^3 + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \quad \text{(Wow!!)}$$

The baby example "grows up" just like you would expect...

Formula: If A is 2×2 with
eigenvalue λ, λ
eigenvector $\underline{v}$
generalized eigenvector $\underline{w}$ then

$\Rightarrow$ $\underline{x}' = A\underline{x}$ has solution

$$\underline{x} = c_1 e^{\lambda t}\underline{v} + c_2 e^{\lambda t}(\underline{w} + t\underline{v})$$

$$\Rightarrow e^{At} = \underbrace{\begin{bmatrix} | & | \\ \underline{v} & \underline{w} \\ | & | \end{bmatrix}\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t}\end{bmatrix}}_{\Psi(t)}\underbrace{\begin{bmatrix} | & | \\ \underline{v} & \underline{w} \\ | & | \end{bmatrix}^{-1}}_{(\Psi(0))^{-1}}$$

EX: $\underline{x}' = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}\underline{x}$ (from beginning of lecture)

has $\lambda = 3,3$ with $\underline{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\underline{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

General solution:

$$\underline{x} = c_1 e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{3t}\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$$= c_1 e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} t \\ -1+2t \end{bmatrix}$$

EX: $\underline{x}' = \begin{bmatrix} 5 & 9 \\ -1 & -1 \end{bmatrix}\underline{x}$ with $\underline{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

has $\lambda = 2,2$ with $\underline{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\underline{w} = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}$

Solution: $\underline{x} = e^{At}\cdot \underline{x}(0)$

$$= \begin{bmatrix} 3 & 0 \\ -1 & 1/3 \end{bmatrix}\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 1/3 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

As a fun computation, we can check why $e^{\lambda t}(\underline{w} + t\underline{v})$ gives a second solution

$\Rightarrow$ It uses the fact that

$$A\underline{w} = \lambda\underline{w} + \underline{v}$$

to mirror the product rule for $\frac{d}{dt}$

$$\frac{d}{dt}(e^{\lambda t}\cdot tv) = \lambda e^{\lambda t}\cdot tv + e^{\lambda t}v$$

$$A(e^{\lambda t}(\underline{w} + t\underline{v})) = e^{\lambda t}(A\underline{w} + tA\underline{v})$$

$$= e^{\lambda t}(\lambda w + v + t\cdot\lambda v)$$

$$\frac{d}{dt}(e^{\lambda t}(\underline{w} + t\underline{v})) = \lambda e^{\lambda t}(\underline{w} + t\underline{v}) + e^{\lambda t}(\underline{v})$$

These two expressions are equal. $\blacksquare$

The 3×3 case is similar.

Given a chain of generalized λ-eigenv

$$\underline{v} \leadsto \underline{w} \leadsto \underline{u}$$

The fundamental solutions are

- $e^{\lambda t}\underline{v}$
- $e^{\lambda t}(\underline{w} + t\underline{v})$
- $e^{\lambda t}\left(\underline{u} + t\underline{w} + \frac{t^2}{2}\underline{v}\right)$

(It is possible to calculate $\exp\left(\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}\right)$ ... but let's skip that.)

EX: $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$ (from earlier)

has $\lambda = 1, 1, 1$

1-eigenvector $\underline{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$-gen eigenv $\underline{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$-gen eigenv $\underline{u} = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$

(1) $$e^{At} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} e^t & te^t & \frac{t^2}{2}e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1/2 \end{bmatrix}^{-1}$$

(2) General Solution to $\underline{x}' = A\underline{x}$

$$\underline{x} = c_1 e^t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c_2 e^t \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right) + c_3 e^t \left(\begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

EX: $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$ (from earlier)

has $\lambda = 2, 2, 2$

2-eigenvectors $\underline{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\underline{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$-gen eigenv. $\underline{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(1) General Solution to $\underline{x}' = A\underline{x}$

$$\underline{x} = c_1 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{2t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right)$$

e^{At} is more complicated to write because we must use $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ as one of our 2-eigenvectors since this is what $\underline{w}$ generalizes... $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ must replace either $\underline{v}_1$ or $\underline{v}_2$

these are 0 because $\underline{w}$ is not a gen² eigenv.

(2) $$e^{At} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} e^{2t} & \boxed{0} & \boxed{0} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix}^{-1}$$